

Abstract

Control charts are widely applied in industrial practice to maintain manufacturing processes in desired operating conditions. Design of control charts aims at finding the best parameters for the operation of chart. In recent years, it has been revealed that the control charts operating with VSI (variable sampling interval) schemes give better performance than with FSI (fixed sampling interval) schemes in the sense of quick response to process shifts. In this paper, a possible combination of design parameters for VSI $\bar{X}$ is considered as a decision-making unit; it is characterized by three attributes: expected loss cost during a production cycle, average run length of process being controlled, and detection power of the control chart designed with the selected parameters. Accordingly, optimal design of control charts can be formulated as a multiple objective decision making (MODM) problem. To solve the MODM problem, a solution procedure on the basis of data envelopment analysis (DEA) is proposed. An industrial application is also presented to illustrate the solution procedure. Finally, comparisons with other models are made to show the effectiveness of the presented model.

Keywords: Statistical process control; Variable sampling interval; Multiple objective decision making; Data envelopment analysis.

1. Introduction

When designing a control chart is discussed, firstly, three parameters sample size ($n$), sampling interval ($h$) and control limits ($k$) should be determined. In doing so, Many people have worked in this area. The most famous Ones is Duncan [1] who suggested the first economic model of $\bar{X}$ control charts to determine these three parameters for processes with single assignable cause. Later, Duncan [2] extended his previous model to multiple assignable causes. In Duncan’s models it
was assumed that the process is not stopped during the search and repair. Another popular model presented by Chiu [3] expressed that when control chart signals production process will be stopped for searching the assignable causes. Also, an important economic model was introduced by Lorenzen and Vance [4] in which the complexity of the model analysis was simplified.

Besides, in order to improve the statistical property of an $\bar{X}$ control chart designed by economic models, Saniga [5] proposed an economic-statistical model by adding constraints of types I and type II errors to the model of Duncan [1]. Del Castillo et al. [6] proposed a model for the design of $\bar{X}$ control chart from a multiple-objective view-point, and an interactive multi-criteria nonlinear optimization algorithm was applied to the model. The aim was to maximize two statistical objective functions and minimize a cost function simultaneously. Furthermore, Chen and Liao [7] formulated optimal design of control charts as a multiple criteria decision-making with regard to some constraints presented by Saniga [5]. Finally, Asadzadeh and Khoshalhan [8] followed the Chen and Liao [7] and extended MODM design of $\bar{X}$ control charts to multiple assignable causes.

Since in traditional control charts parameters $n$, $h$ and $k$ are fixed during the process, the idea of varying each parameter presented with the title of variable or adaptive control charts. One of these charts is variable sampling interval (VSI) that is investigated completely in Reynolds et al. [9]. After that, many people paid attention to these charts to see more aspects of these charts. For example, Bai and Lee [10] followed the Chiu’s model [3] and presented the economic design of a VSI $\bar{X}$ control chart. Yu and Hou [11] optimized the control chart parameters with multiple assignable causes and variable sampling intervals. Lastly, Yu et al. [12] used Duncan’s model and the proposed constraints by Saniga [5] to investigate economic-statistical design of $\bar{X}$ control chart. Also, being inspired by the work of Bai and Lee [10], Chen [13] proposed a cost model which employed the Burr distribution for VSI $\bar{X}$ control charts under normality assumption of process data. In most of the works done before, it is common to consider the existence of only one assignable cause in the process or see only the case of normal data. So, Toosheghanian [14] developed a cost model for processes in which there are multiple assignable causes and also non-normal data.

The problem to select a good control chart design becomes easier if a single criterion, such as the economic criterion is used. However, when several criteria are taken into account for evaluating designs’ performance, the difficulty in comparing the global performance of design becomes evident. A generic design ‘A’ can be easily compared with another design ‘B’ if ‘A’ performs, along all criteria, better than or equal to ‘B’. However, majority comparisons on designs’ performances are difficult to make because design ‘A’ often performs only partially better than design ‘B’. As a result, a tool for optimally selecting the feasible design on control charts is necessary, and data envelopment analysis (DEA) can be helpful for such a purpose when the design’s global performance is defined as the ratio of achieved quality measures (outputs) to cost expenditure (inputs).

Chen and Liao [7] used DEA to solve multi-criteria design of an $\bar{X}$ control chart based on Duncan’s model [1] with a single assignable cause. Concerning the importance of the models with multiple assignable causes and non-normal data, the purpose of this paper is to find the optimum design parameters of VSI $\bar{X}$ control charts using DEA in the presence of multiple assignable causes and non-normal data which satisfy all economic and statistical objectives.
2. Single-Objective Design of VSI $\bar{X}$ Control Charts

2.1. Economic Design of VSI $\bar{X}$ Control Charts

The determination of the VSI $\bar{X}$ control chart parameters by means of minimizing an appropriate cost function is called the economic design. In order to design a VSI $\bar{X}$ control chart with a minimum implementation cost, model (1) is usually used. In this model $E_{LC}$ represents the cost function that is the function of the implementation characteristics of the chart, $n$ is the sample size, $h_1$ and $h_2$ are the first and second sampling intervals, $k_1$ and $k'_1$ are the upper and lower warning limits, and $k_2$ and $k'_2$ are the upper and lower control limits of the VSI $\bar{X}$ control chart.

Minimize $E_{LC}(s)$
Subject to

\begin{align*}
  n & \text{ is a positive integer } \quad \forall \text{ design } s = (n, h_1, h_2, k_1, k_2, k'_1, k'_2) \\
  h_1, h_2, k_1, k_2, k'_1, k'_2 & > 0
\end{align*}

(1)

2.2. Economic-Statistical Design of VSI $\bar{X}$ Control Charts

In order to improve the statistical properties of the control chart, some statistical constraints are added to the economic design to make it an economic-statistical model. Constraints on types I and II errors and also ARL are different methods for this work. By adding these constraints to the economic model, the economic-statistical model is derived as follows:

Minimize $E_{LC}(s)$
Subject to

\begin{align*}
  p(s) & \geq p_L \\
  \alpha(s) & \leq \alpha_L \quad \forall \text{ design } s = (n, h_1, h_2, k_1, k_2, k'_1, k'_2)
\end{align*}

(2)

Usually, economic and economic-statistical designs of the control charts consist of a complex nonlinear cost function which could not be minimized by classical optimization methods. In these situations, metaheuristic algorithms can be used as important and efficient tools.

3. Multiple-Objective Design of VSI $\bar{X}$ Control Charts

For the quality control (QC) manager, who looks after both sides of measurable process cost and quality (likely associated with immeasurable cost), the optimal design issue of control charts to be addressed can be thought as a multi-criteria decision-making problem, the solution of which involves trade-offs and compromise. Therefore, one of the most important steps before decision-making is to identify all criteria, the QC manager may consider, selecting design parameters. In this study, we extend Saniga’s model [5] by adding criteria other than economic criterion to the following multiple objective decision-making (MODM) model:
Maximize $ARL_0(s)$
Maximize $p(s)$
Minimize $E_{LC}(s)$

Subject to

$p(s) \geq p_s$

$\alpha(s) \leq \alpha_c \quad \forall \text{design } s = (n, h_1, h_2, k_1, k_2, k_1^*, k_2^*)$ (3)

For the sake of emphasizing that the values of $ARL_0, p$, and $\alpha$ are affected by $s$, we replace the original symbols $ARL_0, p$, and $\alpha$ with $ARL_0(s), p(s)$, and $\alpha(s)$ hereafter. Essentially, these three objective functions are conflicting from the viewpoint of statistics. We explain it as follows. First, the first two objective functions are conflicting because, in most cases of design control chart, there is a tradeoff between lower false alarm rate (the reciprocal of in-control average run length) and failure detection power when seeking design parameters. Lower false alarm rate will enhance the confidence in the control mechanism, while high failure detection power of control chart will help to improve outgoing quality that relates to the consumer directly. It may be possible to keep false alarm rate low and detection power high simultaneously by increasing sample size $n$. However, it would be expensive, so the third conflicting objective by minimization of cost function, $E_{LC}(s)$, is included.

This research uses a method for seeking the optimal design parameters, which is still based on the MODM model but requires much less computation. It simply regards a set of discrete and finite combinations of design parameters as the solution space. In other words, the foregoing mathematical model aims to find the corresponding objective values of each combination $s$. Thus, choosing a parameter combination by its objective values can be considered as a multi-criteria decision-making (MCDM) problem.

To solve MODM problems, a large number of methods can be used, such as Zionts and Wallenius [15], goal programming [16], global criteria [17], and DEA [18]. Among them, the DEA approach is one of the most powerful and a popular method to optimize the feasible DMUs specifically when measuring the efficiencies of similar units is under consideration. If one defines suitable DMUs for the MODM problem, DEA approach may be employed to achieve the optimum solutions. In the proposed MODM problem, a combination $s$ is regarded as a decision-making unit (DMU) with its attributes of expected loss cost $E_{LC}(s)$, in-control average run length $ARL_0(s)$, and detection power, $p(s)$. Subsequently, DEA is used to optimize the feasible combinations of design parameters by measuring their efficiencies that is the ratio of achieved quality (outputs) to cost expenditure (inputs).

4. Data Envelopment Analysis (DEA)

Data envelopment analysis (DEA) is a linear programming based technique for measuring and comparing the relative efficiency of a set of competing decision-making units (DMU) where the presence of multiple-inputs–multiple-outputs makes the comparisons difficult [19]. The relative efficiency of the ‘multiple-inputs–multiple-outputs’ in DMU is typically defined as engineering like
ratio (weighted sum of the DMU’s outputs divided by weighted sum of the DMU’s inputs), i.e. for the generic rth DMU:

\[
E_r = \frac{\text{weighted sum of outputs}}{\text{weighted sum of inputs}}
\]  

(4)

So, if a relative efficiency wants to have higher performance, the input data of ratio must have lower value and the output data of ratio must have higher value. In this paper, we utilize the characteristic of DEA, that is, lower input value with higher output value gets higher relative efficiency of DMU, to solve the issue of control chart design. When applying it, \( s = (n, h_1, h_2, k_1, k_2, k'_1, k'_2) \) is deemed as a DMU with its single input is the expected cost loss cost during a production cycle \( E_{LC}(s) \), and two outputs: \( \text{ARL}_s(s) \) and \( p(s) \).

The general efficiency measure used by DEA is summarized by Eq. (5)

\[
E_s(s) = \frac{\sum O_y(s)w_{ry}}{\sum I_x(s)w_{rx}}
\]  

(5)

where

- \( E_s(s) \) is the efficiency measure of control chart design \( s \), using the weights of assessed design \( r \);
- \( O_y(s) \) are the values of output \( y \) for design \( s \);
- \( I_x(s) \) are the values of input \( x \) for design \( s \);
- \( w_{ry} \) are the most favorable weights assigned to design \( r \) for output \( y \);
- \( w_{rx} \) are the most favorable weights assigned to design \( r \) for input \( x \).

In evaluating the relative efficiency of a DMU it is necessary to determine how the weights are to be established. In contrast to other techniques such as statistical method, which give a single common set of weights for each DMU, DEA allows each DMU to choose the set of weights that permits it to appear in the best light. That is, the most favorable set of weights, for each DMU, is adopted to compare with other competing DMU. It improves the shortcoming that assesses the relative efficiency of each DMU by adopting a single common set of weights for each DMU may be not easy and correct in many cases. So, following the spirit of DEA, a specific design with parameters \((n, h_1, h_2, k_1, k_2, k'_1, k'_2)\) that leads to slightly higher cost and extremely high quality will be provided a big weight for its quality item to appear in the best light, and this is why DEA is appropriate to be applied to solve the problem of selecting design parameters in a multi-criteria sense.

To decide the optimal set of weights for the \( r \)th parameter combination (the \( r \)th DMU), many mathematical models have been developed in literature. Within them the CCR model developed by Charnes, Cooper, and Rhodes [20] is most popular. The objective in CCR model is to maximize the relative efficiency value of an assessed design \( r \) from among a reference set of design \( s \), by selecting the optimal weights associated with the input and output measures. The maximum relative efficiencies are constrained to 1. The formulation is represented in expression (6).
Maximize \( E_r(s) = \frac{\sum y O_y(s)w_r}{\sum y I_y(s)w_r} \)

Subject to
\[
E_r(s) \leq 1 \quad \forall \text{ other designs } s
\]
\( w_r, w_r > 0 \)

This non-linear programming formulation (6) is equivalent to the following linear programming (LP) formulation by setting its denominator equal to 1 and maximizing its numerator.

Maximize \( E_r(r) = \sum y O_y(r)w'_r \)

Subject to
\[
\sum y I_y(r)w'_r = 1
\]
\[
E_r(r) \leq 1 \quad \forall \text{ other designs } s
\]
\( w'_r, w'_r > 0 \)

The result of formulation (7) is an optimal efficiency value, \( E^*_r(r) \), that is at most 1. If \( E^*_r(r) = 1 \), then no other design is more efficient than design \( r \) under its own weights. That is, \( E^*_r(r) = 1 \) has design \( r \) on the optimal frontier and is not dominated by other design. If \( E^*_r(r) < 1 \), then design \( r \) does not lie on the optimal frontier, and there is at least one other design that is more efficient under the optimal set of weights determined by (7). The formulation (9) is employed for each design to calculate design’s efficiency with respect to its own optimal set of weights. As for more details about the theory, applications, and software package of DEA, please refer to Charnes, Cooper, and Seiford [21].

Applying DEA technique, the three performance values \( (E_{LC}(s), \text{ARL}_0(s), \text{p}(s)) \) of each specific control chart design will be combined to a ratio form, which is called relative efficiency in terms of DEA.

5. Solution Method
In this paper, we have employed the four-step algorithm introduced by Chen and Liao [7]. They applied this procedure to their multi-criteria decision-making model with one assignable cause cost function.

Once process parameters or cost parameters are estimated, the values of \( E_{LC}(s), \text{ARL}_0(s), \text{p}(s) \) for each potential combination \( s \) will be calculated, respectively. The calculation work can be facile by means of Microsoft Excel.

According to their attributes, a solution procedure with the help of DEA software package is proposed to evaluate and compare their performance in terms of the ratio-quality measures to cost expenditure. We summarize it as follows;
Step 1. *Determining the potential solution.* The suitable scope for each design parameter should be initially confined to form a set ($\Omega$) of potential parameter combinations $s$. For instance, the scope of sample size $n$ may be confined as from 1 to 30, increased by 1. Greatly large sample size is not taken into account due to high inspection expenditure. Similarly, the scope of sampling time intervals may be set as from 0.1 to 4, increased by 0.1, while the scope for width of control and warning limits are from 0.1 to 3 in terms of standard deviation increased by 0.1.

Step 2. *Leaching process.* Leach unattractive elements of $\Omega$ by the quality constraints $\alpha(s) \leq \alpha_U$ and $p(s) \geq p_L$. The remainders after leaching process are separately collected into a set $Q_n$ by their sample size $n$.

Step 3. *Partial optimization.* Remain the elements with Pareto optimality for each subset $Q_n$. A solution $s$ with Pareto optimization in a set $Q_n$ means that there is no other solution in the same set such that $s$ is dominated in terms of statistical properties and cost.

Step 4. *Global optimization.* Merge all the remainders into a set $W$ and select the elements with highest relative efficiency among $W$. The selected elements will afford to DM to make final decision.

6. Application of the Model

In this section, a numerical example is presented to illustrate the solution procedure for MODM design of the VSI $X$ control chart. The data set provided in Bai and Lee [10] and Yu and Hou [11] is corrected and applied for this study to show the application of the suggested model.

Assume that there are seven types of assignable causes in a production process. Based on this information we have $C_1=10$, $C_2=0.50$, $C_3=0.1$, and $\lambda=0.01$. The parameter $\lambda_m$ is the average occurrence rate of the $m$th assignable cause (per unit time). Therefore, the ratio $\lambda_m/\lambda$ (where $\lambda = \sum \lambda_m$) is the conditional probability for the $m$th assignable cause occurrence while the assignable cause occurs. The shift $\delta_m \sigma$ in the process mean is considered when the $m$th assignable cause occurs.

All the information about data and parameters $W_m$, $D_m$, $\delta_m$, $\lambda_m$ and $M_m$ can be found in Table 1.

<table>
<thead>
<tr>
<th>Item</th>
<th>$\lambda_m$</th>
<th>$\delta_m$</th>
<th>$D_m$</th>
<th>$M_m$</th>
<th>$W_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00225</td>
<td>1.0</td>
<td>0.495</td>
<td>14.339</td>
<td>49.46</td>
</tr>
<tr>
<td>2</td>
<td>0.00175</td>
<td>1.5</td>
<td>0.385</td>
<td>42.108</td>
<td>38.25</td>
</tr>
<tr>
<td>3</td>
<td>0.00151</td>
<td>1.8</td>
<td>0.332</td>
<td>72.528</td>
<td>33.16</td>
</tr>
<tr>
<td>4</td>
<td>0.00136</td>
<td>2.0</td>
<td>0.300</td>
<td>100.000</td>
<td>30.00</td>
</tr>
<tr>
<td>5</td>
<td>0.00123</td>
<td>2.2</td>
<td>0.271</td>
<td>133.532</td>
<td>27.15</td>
</tr>
<tr>
<td>6</td>
<td>0.00106</td>
<td>2.5</td>
<td>0.234</td>
<td>194.470</td>
<td>23.36</td>
</tr>
<tr>
<td>7</td>
<td>0.00083</td>
<td>3.0</td>
<td>0.182</td>
<td>315.149</td>
<td>18.20</td>
</tr>
</tbody>
</table>
It is also assumed that the set of collected data has non-normal distribution with following sample statistics:
\[ \hat{\mu}_x = 50.42, \hat{S}_x = 5.68, \hat{\alpha}_1 = 1.4322, \hat{\alpha}_4 = 7.3558 \]

To play the roles of excellent manufacturer and supplier—the company vision, the QC manager was asked to simultaneously regard the aspects of operation cost and product quality on designing control charts.

The estimated values for \( \alpha_3, \alpha_4 \), and considering the tables provided by Burr [22], we can conclude that a Burr distribution with parameters \( c=2 \) and \( k=4 \) is a suitable choice.

With the proposed model and solution procedure, the optimal values of \( (n, h_1^*, h_2^*, k_1^*, k_2^*, k_1'^*, k_2'^*) \) found by evaluating a wide range of possible solutions are determined through the following steps:

Firstly, large numbers of infeasible solutions are eliminated by applying the statistical constraints \( \alpha_U = 0.1 \) and \( p_L = 0.90 \). Then, the remainders are classified into different sets by their sample sizes. The non-dominated solutions are selected for each group and sent for the final optimization process. In the end, DEA is employed for evaluating the efficiency of these solutions, and the solution with largest efficiency score is selected as the optimum for design parameters \( (n, h_1, h_2, k_1, k_2, k_1', k_2') \). Due to the complicated VSI policy and non-normality with multi-assignable cause cost function, all calculations have been facilitated by Excel software. In addition, to evaluate and compare the efficiencies of DMUs, Microsoft Excel with XlDEA has been implemented. The results of employing DEA show that only 3 combinations of design parameters have received efficiency score 1 and thus are offered to the DM for final selection. These combinations are shown in Table 2. The DM may choose the first combination if low cost is beneficial for him. Similarly, if he is much more interested in the outgoing quality, then the second and third combinations with large average run length and detection power may be the final choice.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( h_1 )</th>
<th>( h_2 )</th>
<th>( k_1 )</th>
<th>( k_2 )</th>
<th>( k_1' )</th>
<th>( k_2' )</th>
<th>Cost</th>
<th>Power</th>
<th>ARL</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td>1.3000</td>
<td>2.8000</td>
<td>2.2000</td>
<td>2.6000</td>
<td>2.1000</td>
<td>2.5000</td>
<td>2.1386</td>
<td>0.9432</td>
<td>114.9425</td>
</tr>
<tr>
<td>25</td>
<td>1.4000</td>
<td>2.7000</td>
<td>2.3000</td>
<td>2.4000</td>
<td>2.3000</td>
<td>2.3000</td>
<td>2.2957</td>
<td>0.9704</td>
<td>222.0528</td>
</tr>
<tr>
<td>29</td>
<td>1.4000</td>
<td>2.9000</td>
<td>2.5000</td>
<td>2.7000</td>
<td>2.4000</td>
<td>2.7000</td>
<td>2.4498</td>
<td>0.9793</td>
<td>256.4102</td>
</tr>
</tbody>
</table>

7. Comparisons with Economic and Economic-Statistical Designs

A comparison between the economic and economic-statistical designs with the proposed multi-objective design is shown in Table 3. It reveals that the sample sizes are 2 and 3 for pure economic and economic-statistical designs. While, sample sizes of MODM design are 23, 25 and 29, respectively. For MODM design, the expected loss cost during a cycle in contrast to economic design increase about 28.9% [(2.1386−1.6589)/ 1.6589], 38.4% [(2.2957−1.6589)/ 1.6589], and 47.7% [(2.4498−1.6589)/ 1.6589], respectively. Corresponding increase in contrast to economic-statistical design are 9.9% [(2.1386−1.9462)/ 1.9462], 18% [(2.2957−1.9462)/ 1.9462], and 25.9%
A similar conclusion can be made for test power and also ARL of the control chart. Sampling intervals increase in the multi-objective design with compared to the economic and economic-statistical designs. Note that in economic-statistical or multi-objective designs, the false alarms (α) and test powers are all in desired limits. It indicates that statistical performance of control charts can be improved substantially with a slight increase in the cost by using economic-statistical or multi-objective designs.

Table 3. Comparison of economic and economic-statistical designs with MODM design

<table>
<thead>
<tr>
<th>Design</th>
<th>n</th>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$k'_1$</th>
<th>$k'_2$</th>
<th>Cost</th>
<th>Power</th>
<th>ARL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Economic</td>
<td>2</td>
<td>0.0056</td>
<td>2.1959</td>
<td>2.9961</td>
<td>0.0039</td>
<td>2.8983</td>
<td>2.3499</td>
<td>1.6589</td>
<td>0.5197</td>
<td>3.3047</td>
</tr>
<tr>
<td>Economic-Statistical</td>
<td>3</td>
<td>0.0070</td>
<td>2.1396</td>
<td>2.6377</td>
<td>0.0489</td>
<td>2.9315</td>
<td>2.2770</td>
<td>1.9462</td>
<td>0.9117</td>
<td>23.3590</td>
</tr>
<tr>
<td>MODM</td>
<td>23</td>
<td>1.3000</td>
<td>2.8000</td>
<td>2.2000</td>
<td>2.6000</td>
<td>2.1000</td>
<td>2.5000</td>
<td>2.1386</td>
<td>0.9432</td>
<td>114.9425</td>
</tr>
<tr>
<td>MODM</td>
<td>25</td>
<td>1.4000</td>
<td>2.7000</td>
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<td>2.3000</td>
<td>2.3000</td>
<td>2.2957</td>
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<tr>
<td>MODM</td>
<td>29</td>
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<td>2.7000</td>
<td>2.4498</td>
<td>0.9793</td>
<td>256.4102</td>
</tr>
</tbody>
</table>

8. Conclusions

In this paper, a model for the design of a VSI $\bar{X}$ control chart from a multiple criteria viewpoint has been presented. With this model, a set of design parameters ($n$, $h_1$, $h_2$, $k_1$, $k_2$) for the VSI $\bar{X}$ control chart is chosen based on data envelopment analysis and provides the QC manager a variety of choices to arrive at the requirement of long run quality of product or minimal cost concurrently. A clear advantage of the proposed methodology is that only the costs of sampling, which normally are the easiest to estimate, are needed. The user does not need to specify the cost of false alarms and of running the process out of control. An industrial case is presented to illustrate the solution procedure. Comparisons with pure economic and economic-statistical designs and multi-objective model have been made, which reveal that our multi-objective model can overcome the drawbacks of mentioned models. An interesting research area for future research involves investigating and comparing the use of other methods except DEA to solve the multi-objective model. In addition, multi-objective design of VSI $\bar{X}$ control chart under correlation may be considered as the other potentially useful area for future research. Moreover, a different failure mechanism, such as Weibull distribution, will be next important issue for other researchers.

References


